# Equiconvergence of Some Sequences of Complex Interpolating Rational Functions*

## (Quantitative Estimates and Sharpness)

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## 1. INTRODUCTION

In his classic book on interpolation and approximation, Walsh [1] has shown that approximation in the sense of least squares by polynomials is intimately connected with Taylor series and he suggested that "approximation in the sense of least squares by more general rational functions may also be connected with interpolation in points related to the poles of the rational functions." A number of his theorems [1, Chapter IX] justify this assertion. Recently, Saff and Sharma [2] took the cue and proved a theorem which further supplements the above statement of Walsh.

More precisely, let $A_\rho$, $\rho > 1$, be the class of functions analytic in $|z| < \rho$ but not in $|z| \leqslant \rho$. For a given integer $m \geqslant -1$ and for $\sigma > 1$ and $f \in A_\rho$, let $R_{n+m,n}(z, f)$ be a rational function of the form

$$R_{n+m,n}(z, f) := B_{n+m,n}(z, f)/(z^n - \sigma^n), \qquad (B_{n+m,n}(z, f) \in \pi_{n+m}), \quad (1.1)$$

which interpolates $f$ in the $(n+m+1)$th roots of unity. For $v = 0, 1, 2, \ldots$ let (cf. [2, (3.7)])

$$f(z) = \sum_{v=0}^{\infty} \left\{ \frac{\beta_{n,m}(z)}{\alpha_{n,m}(z)} \right\}^{v} r_{n+m,n}(z, f, v), \quad (1.2)$$

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where

$$\alpha_{n,m}(z) = 1 - z^{m+1}\sigma^{-n}, \qquad \beta_{n,m}(z) = z^{m+1}(z^n - \sigma^{-n}),$$

$$r_{n+m,n}(z, f, v) = \frac{P_{n+m,n}(z, f, v)}{z^n - \sigma^n}, \qquad P_{n+m}(z, f, v) \in \pi_{n+m}. \tag{1.3}$$

For any integer $l \geq 1$, we set

$$\Delta_{l,n,m}^{\sigma}(z, f) := R_{n+m,n}(z, f) - \sum_{v=0}^{l-1} r_{n+m,n}(z, f, v). \tag{1.4}$$

Saff and Sharma [2] proved

THEOREM A.   *Let $\rho > 1$, $\sigma > 1$, and an integer $m \geq -1$ be fixed. If $f \in A_\rho$ and if $l$ is a given positive integer, then*

$$\lim_{n \to \infty} \Delta_{l,n,m}^{\sigma}(z, f) = 0 \quad \begin{cases} \text{for } |z| < \rho^{1+l}, & \text{if } \sigma \geq \rho^{1+l}, \\ |z| < \sigma \text{ and } |z| > \sigma, & \text{if } \sigma < \rho^{1+l}, \end{cases} \tag{1.5}$$

*the convergence being uniform and geometric on compact subsets of these regions. Moreover, the result is best possible in the sense that for each $z$ with $|z| = \rho^{1+l}$ and $\sigma \geq \rho^{1+l}$, there is an $\tilde{f} \in A_\rho$ for which (1.5) does not hold.*

*Remark* 1.1.   $r_{n+m,n}(z, f, 0)$ is the rational function which, besides interpolating $f$ in the $n+m+1$ zeros of $\beta_{n,m}(z)$, also minimizes the integral $\int_{|z|=1} |f(z) - (p(z)/(z^n - \sigma^n))|^2 |dz|$ over all polynomials $p(z) \in \pi_{n+m}$.

Theorem A may be looked upon as a direct theorem in the Walsh equiconvergence theory. In the present paper, we derive some quantitative results for the sequence $\{\Delta_{l,n,m}^{\sigma}(z, f)\}_{n=1}^{\infty}$. To achieve our goal, we are motivated by a recent paper of V. Totik [3] on complex interpolating polynomials.

## 2. STATEMENT OF NEW RESULTS

Let $R$ be a positive real number different from $\sigma$. For $\sigma > \rho$, set

$$F_l(R, \sigma) := \varlimsup_{n \to \infty} (\max_{|z| = R} |\Delta_{l,n,m}^{\sigma}(z, f)|)^{1/n} \tag{2.1}$$

and

$$K_l(R, \sigma, \rho) := \rho^{-l} \max\left(\min\left\{\frac{R}{\rho}, \frac{\sigma}{\rho}\right\}, 1\right). \tag{2.2}$$

Then we have the following result:

THEOREM 2.1. *Let* $1 < \rho < \sigma$ *and let* $R > 0$ *be a real number different from* $\sigma$. *If* $f \in A_\rho$ *then*

$$F_l(R, \sigma) = K_l(R, \sigma, \rho), \qquad (2.3)$$

*where* $K_l(R, \sigma, \rho)$ *and* $F_l(R, \sigma)$ *are given by* (2.2) *and* (2.1), *respectively.*

*Remark* 2.1. For fixed $\rho$ and $\sigma$, the value of the function $K_l(R, \sigma, \rho)$ can be described as

$$K_l(R, \sigma, \rho) = \begin{cases} \sigma \rho^{-l-1}, & \text{if } R > \sigma \\ R \rho^{-l-1}, & \text{if } \rho \leqslant R < \sigma \\ \rho^{-l}, & \text{if } 0 < R < \rho. \end{cases} \qquad (2.4)$$

The relation (2.3) does not hold when $\sigma = \rho$. For this consider the following example:

EXAMPLE 2.1. Let $f(z) = (z - \rho)^{-1}$. Then (cf. [2, (3.15)]),

$$\Delta_{l,n,m}^\sigma(z, f) = \frac{(\rho^n - \sigma^n)\{\alpha_{n,m}(z)\,\beta_{n,m}(\rho) - \alpha_{n,m}(\rho)\,\beta_{n,m}(z)\}}{(z^n - \sigma^n)(z - \rho)(\rho^{n+m+1} - 1)\,\alpha_{n,m}(\rho)} \left(\frac{\alpha_{n,m}(\rho)}{\beta_{n,m}(\rho)}\right)^l$$

which is identically zero except at $z = \rho$ if $\sigma = \rho$.

Also, for $R = 0$, Theorem 2.1 is no longer true. We shall justify this statement in the next section (see Remark 3.2).

If $\omega$ is any fixed primitive root of unity, then $z = \sigma\omega$ is a singular point of the function $\Delta_{l,n,m}^\sigma(z, f)$ for infinitely many $n$'s (see Example 2.1). Because of this, we have excluded he case $R = \sigma$ in the relation (2.3).

It may be noted that $K_l(R, \sigma, \rho) \neq K_l(R, \sigma, \rho')$ if $\rho \neq \rho'$. Thus as in [3, Corollary 1] Theorem 2.1 gives

COROLLARY 2.1. *Let* $\sigma$ *and* $\rho'$ *be fixed numbers with* $\sigma > \rho' > 1$ *and let* $f(z)$ *be an analytic function in* $|z| < \rho'$. *If for any fixed integer* $l \geqslant 1$ *and any real number* $\rho$, $\rho' \leqslant \rho < \sigma$, *the relation* (2.3) *holds for some* $R > 0$, *then* $f \in A_\rho$.

Our next concern is to study the pointwise behaviour of $\Delta_{l,n,m}(z, f)$ in the complex plane with some exceptions. If we set

$$Y_1 := \{z : \rho < |z|, |z| \neq \sigma\} \quad \text{and} \quad Y_2 = \{z : |z| < \rho\},$$

then we shall prove

THEOREM 2.2.   *Let* $l \geqslant 1$, $m \geqslant -1$ *be integers and* $\sigma \geqslant \rho^2$. *For each* $f \in A_\rho$, $1 < \rho < \infty$, *we have*

$$\varlimsup_{n \to \infty} |\Delta^\sigma_{l,n,m}(z, f)|^{1/n} = K_l(|z|, \sigma, \rho), \qquad z \in Y_j \qquad (2.5)$$

*except at most at* $l - [j/2]$ *points in* $Y_j$ $(j = 1, 2)$, *where* $K_l(|z|, \sigma, \rho)$ *is given by* (2.4).

*Remark* 2.2.   From the above theorem we note that if $\sigma > \rho^{l+1}$, then the sequence $\{\Delta^\sigma_{l,n,m}(z, f)\}_{n=1}^\infty$, $f \in A_\rho$, cannot be bounded at more than $l$ points in the region $Z = \{z : \sigma \neq |z| > \rho^{l+1}\}$. This gives an analogue of a result of Saff and Varga [4] on the sharpness of some equiconvergence results for interpolating polynomials.

The next result shows that in some sense Theorem 2.2 cannot be improved.

THEOREM 2.3.   *Let* $\rho > 1$, $\sigma \geqslant \rho^2$, *and let integers* $l \geqslant 1$, $m \geqslant -1$ *be fixed. Given any set* $\{z_k\}$ *of* $l - [j/2]$ *distinct points in the region* $Y_j$ $(j = 1, 2)$, *there exist rational functions* $f_j \in A_\rho$ $(j = 1, 2)$ *for which*

$$\varlimsup_{n \to \infty} |\Delta^\sigma_{l,n,m}(z_k, f_j)|^{1/n} < K_l(|z_k|, \sigma, \rho) \qquad (j = 1, 2)$$

*for every* $k = 1, 2, \dots, l - [j/2]$.

Finally, we remark that Theorems 2.1–2.3 hold when $m < -1$.

## 3. REPRESENTATION OF $\Delta^\sigma_{l,n,m}(z, f)$

The proofs of the above theorems will be based on a representation of $\Delta^\sigma_{l,n,m}(z, f)$ in terms of the Taylor coefficients of $f$ which is given by (3.15) in Lemma 3.2.

In order to establish (3.15), we recall hat (cf. [2, Corollary 3.2])

$$(z^n - \sigma^n)\, \Delta^\sigma_{l,n,m}(z, f) = \sum_{v=l}^\infty P_{n+m,n}(z, f, v), \qquad (3.1)$$

where

$$P_{n+m,n}(z, f, v) = \frac{1}{2\pi i} \int_\Gamma \frac{(t^n - \sigma^n)\, f(t)\, K_{n,m}(z, t)}{\alpha_{n,m}(t)\, \beta_{n,m}(t)} \left(\frac{\alpha_{n,m}(t)}{\beta_{n,m}(t)}\right)^v dt \qquad (3.2)$$

with

$$K_{n,m}(z, t) := \frac{\alpha_{n,m}(z)\, \beta_{n,m}(t) - \alpha_{n,m}(t)\, \beta_{n,m}(z)}{t - z}. \qquad (3.3)$$

Here $\Gamma$ is a circle $|t| = \rho_1$, $1 < \rho_1 < \rho$. From the definitions of $\beta_{n,m}(t)$ and $\alpha_{n,m}(t)$ in (1.3), we easily see that

$$(\beta_{n,m}(t))^{-\nu-1} = t^{-(\nu+1)(n+m+1)} \sum_{s=0}^{\infty} \binom{\nu+s}{s} \sigma^{-ns} t^{-ns}$$

$$(\alpha_{n,m}(t))^{\nu-1} = \sum_{j=0}^{\nu-1} (-1)^j \binom{\nu-1}{j} \sigma^{-nj} t^{(m+1)j} \tag{3.4}$$

$$K_{n,m}(t) = t^{n+m} \sum_{k=0}^{n+m} b_{n,k}(z, \sigma) t^{-k} z^k,$$

where

$$b_{n,k}(z, \sigma) := \begin{cases} 1 - \sigma^{-n} z^{m+1}, & 0 \leqslant k \leqslant n-1 \\ 1 - \sigma^{-n} z^{-n}, & n \leqslant k \leqslant n+m. \end{cases} \tag{3.5}$$

Using (3.2)–(3.5), we see after some elementary computation that

$$P_{n+m,n}(z, f, \nu) = \sum_{j=0}^{\nu-1} (-1)^j \binom{\nu-1}{j} \sigma^{-nj} \sum_{s=0}^{\infty} \binom{\nu+s}{s} \sigma^{-ns} S_{n+m}(z, j, s),$$

$$\tag{3.6}$$

where

$$S_{n+m}(z, j, s) = \sum_{k=0}^{n+m} (a_{N(k)-n} - \sigma^n a_{N(k)}) b_{n,k}(z, \sigma) z^k \tag{3.7}$$

with $N(k) := \nu(n+m+1) - j(m+1) + ns + k$. Setting

$$I_{\nu,k,n} \equiv I_{\nu,k,n}(f) := \sum_{j=0}^{\nu-1} (-1)^j \binom{\nu-1}{j} \sigma^{-nj} \sum_{s=0}^{\infty} \binom{\nu+s}{s} \sigma^{-ns} a_{N(k)}, \tag{3.8}$$

it follows from (3.5)–(3.7) that

$$P_{n+m,n}(z, f, \nu) = \sum_{k=0}^{n+m} (I_{\nu,k-n,n} - \sigma^n I_{\nu,k,n}) z^k$$

$$- \sigma^{-n} \sum_{k=0}^{n-1} (I_{\nu,k-n,n} - \sigma^n I_{\nu,k,n}) z^{k+m+1}$$

$$- \sigma^{-n} \sum_{k=0}^{m} (I_{\nu,k,n} - \sigma^n I_{\nu,k+n,n}) z^k. \tag{3.9}$$

LEMMA 3.1. *Let* $l \geqslant 1$ *and* $1 < \rho_1 < \rho$. *Using the above notation the following inequalities hold,*

$$I_{l,k,n} = a_{l(n+m+1)+k} + O(\sigma^{-n}\rho_1^{-ln-k})$$

$$\sum_{v=l+1}^{\infty} I_{v,k,n} = a_{(l+1)(n+m+1)+k} + O(\sigma^{-n}\rho_1^{-(l+1)n-k}) \qquad (0 \leqslant k \leqslant n+m),$$

$$(3.10)$$

*where the constants depend on* $\sigma$, $\rho$, $\rho_1$, $f$, *and* $m$.

*Proof.* (i) From the Cauchy integral formula and the definition of $N(k)$ we can rewrite (3.8) for $v = l$, and after some simplification we obtain

$$I_{l,k,n} = \frac{1}{2\pi i} \int_\Gamma \frac{f(t)}{t^{l(n+m+1)+k+1}} \left( \frac{(1-\sigma^{-n}t^{m+1})^{l-1}}{(1-\sigma^{-n}t^{-n})^{l+1}} \right) dt. \qquad (3.11)$$

Since $\sigma > 1$ and $|t| = \rho_1 > 1$, we have

$$\frac{(1-\sigma^{-n}t^{m+1})^{l-1}}{(1-\sigma^{-n}t^{-n})^{l+1}} = 1 + \frac{(1-\sigma^{-n}t^{m+1})^{l-1} - (1-\sigma^{-n}t^{-n})^{l+1}}{(1-\sigma^{-n}t^{-n})^{l+1}}$$

$$= 1 + O(\sigma^{-n})$$

so that from (3.11) we obtain the first relation in (3.10).

(ii) Summing up the relation (3.8) over $v$ from $l+1$ to $\infty$, we obtain (cf. (3.11))

$$\sum_{v=l+1}^{\infty} I_{v,k,n} = \frac{1}{2\pi i} \int_\Gamma \frac{f(t)}{t^{(l+1)(n+m+1)+k+1}} \left( \frac{(1-\sigma^{-n}t^{m+1})^l}{(1-\sigma^{-n}t^{-n})^{l+2}} \right)$$

$$\times \frac{t^{m+1}(t^n - \sigma^n)}{t^{n+m+1} - 1} dt.$$

With an argument similar to the above one, we can easily derive the second relation in (3.10) which completes the proof. ∎

LEMMA 3.2. *Let* $\rho > \rho_1 > 1$ *and* $l$ *be a positive integer. If* $\sigma > \rho$ *then* (cf. (3.9))

$$P_{n+m,n}(z, f, l) = \sum_{k=0}^{n+m} (a_{l(n+m+1)-n+k} - \sigma^n a_{l(n+m+1)+k})z^k$$

$$+ O\left( \rho_1^{-ln} \sum_{k=0}^{n+m} \left( \frac{|z|}{\rho_1} \right)^k \right). \qquad (3.12)$$

*Proof.* From Lemma 3.1, we get

$$I_{l,k-n,n} - \sigma^n I_{l,k,n} = a_{l(n+m+1)-n+k} - \sigma^n a_{l(n+m+1)+k}$$
$$+ O(\rho_1^{-(l-1)n-k}\sigma^{-n}) + O(\rho_1^{-ln-k}). \qquad (3.13)$$

Since $\overline{\lim}_{r\to\infty} |a_r|^{1/r} = \rho^{-1}$, we have $|a_r| = O(\rho_1^{-r})$, $1 < \rho_1 < \rho$. If $\sigma > \rho$, it can be deduced easily from (3.11) that

$$I_{l,k-n,n} - \sigma^n I_{l,k,n} = O(\sigma^n \rho_1^{-ln-k}). \qquad (3.14)$$

Thus (3.12) follows from (3.9) and (3.13). ∎

Now we can represent $\Delta_{l,n,m}^{\sigma}(z, f)$ *explicitly* in terms of the Taylor coefficients of $f(z)$ as below.

LEMMA 3.3. *Let $m \geqslant -1$ and $l \geqslant 1$ be fixed integers and let $f \in A_\rho$, $1 < \rho < \infty$. If $\sigma > \rho$, then for every $\varepsilon > 0$ the following relation holds,*

$$(z^n - \sigma^n)\Delta_{l,n,m}^{\sigma}(z, f) = \sum_{k=0}^{n+m} \{a_{l(n+m+1)-n+k} - \sigma^n a_{l(n+m+1)+k}\} z^k$$
$$+ G_n(\rho_1, \sigma, z), \qquad (3.15)$$

*where $\rho_1 = \rho - \varepsilon$ and*

$$G_n(\rho_1, \sigma, z) = O\left\{\sigma^n \rho_1^{-(l+1)n} \sum_{k=0}^{n+m} \left(\frac{|z|}{\rho_1}\right)^k\right\}. \qquad (3.16)$$

*Proof.* We can rewrite (3.1) as

$$(z^n - \sigma^n)\Delta_{l,n,m}^{\sigma}(z, f) = P_{n+m,n}(z, f, l) + \sum_{v=l+1}^{\infty} P_{n+m,n}(z, f, v). \qquad (3.17)$$

If we sum up the relation (3.9) over $v$ from $l+1$ to $\infty$, and then use the second formula in (3.10), we obtain after some computation

$$\sum_{v=l+1}^{\infty} P_{n+m,n}(z, f, v) = O\left(\sigma^n \rho_1^{-(l+1)n} \sum_{k=0}^{n+m} \left(\frac{|z|}{\rho_1}\right)^k\right) \qquad (3.18)$$

which dominates $O(\rho_1^{-ln} \sum_{k=0}^{n+m} (|z|/\rho_1)^k)$ when $\sigma > \rho$. Hence (3.15) follows from (3.17), (3.12), and (3.18). ∎

*Remark* 3.1. We pointed out in Section 2 that, in general, Theorem 2.1 is not true for $R = |z| = 0$. For this, notice that (3.15) and (3.16) give

$$-\sigma^n \Delta_{l,n+m}^{\sigma}(0, f) = a_{l(n+m+1)-n} - \sigma^n a_{l(n+m+1)} + O\{\sigma^n \rho_1^{-(l+1)n}\}. \qquad (3.19)$$

It may happen for some $f \in A_\rho$ that $a_{l(n+m+1)-n} = a_{l(n+m+1)} = 0$, for all $n$, in which case $F_l(0, \sigma) \leqslant \rho^{-(l+1)}$.

The proof of Theorem 2.2 essentially depends on the representation (3.21) for the function $H_n(z)$ given by

$$H_n(z) := \sigma \, \varDelta_{l,n,m}^\sigma(z, f) - \frac{z^{n+1} - \sigma^{n+1}}{z^n - \sigma^n} z^l \varDelta_{l,n+1,m}^\sigma(z, f). \qquad (3.20)$$

LEMMA 3.4.   *Let* $l \geqslant 1$, $1 < \rho_1 < \rho$, *and* $\sigma \geqslant \rho^2$. *If* $f \in A_\rho$ *then*

$$(z^n - \sigma^n) H_n(z) = -\sigma^{n+1} \sum_{k=0}^{l-1} a_{l(n+m+1)+k} z^k$$

$$+ \sigma^{n+1} \sum_{k=0}^{l} a_{(l+1)(n+m+1)+k} z^k + G_{n+1}(\rho_1, \sigma, z), \qquad (3.21)$$

*where* $G_n(\rho_1, \sigma, z)$ *is given by* (3.16).

*Proof.* From (3.15), after using the fact $a_{l(n+m+1)-n+k} = O(\rho_1^{-(l-1)n-k})$, $0 \leqslant k \leqslant n+m$, we can write

$$(z^n - \sigma^n) \varDelta_{l,n,m}^\sigma(z, f) = -\sigma^n \sum_{k=0}^{n+m} a_{l(n+m+1)+k} z^k + O\left\{\rho_1^{-(l-1)n} \sum_{k=0}^{n+m} \left(\frac{|z|}{\rho_1}\right)^k\right\}$$

$$+ O\left\{\sigma^n \rho_1^{-(l+1)n} \sum_{k=0}^{n+m} \left(\frac{|z|}{\rho_1}\right)^k\right\}.$$

Since $\sigma \geqslant \rho^2$, it follows that

$$(z^n - \sigma^n) \varDelta_{l,n,m}^\sigma(z, f) = -\sigma^n \sum_{k=0}^{n+m} a_{l(n+m+1)+k} z^k$$

$$+ O\left\{\sigma^n \rho_1^{-(l+1)n} \sum_{k=0}^{n+m} \left(\frac{|z|}{\rho_1}\right)^k\right\}. \qquad (3.22)$$

Similarly,

$$(z^{n+1} - \sigma^{n+1}) \varDelta_{l,n+1,m}^\sigma(z, f) = -\sigma^{n+1} \sum_{k=0}^{n+m+1} a_{l(n+m+2)+k} z^k$$

$$+ O\left\{\sigma^n \rho_1^{-(l+1)n} \sum_{k=0}^{n+m+1} \left(\frac{|z|}{\rho_1}\right)^k\right\}. \qquad (3.23)$$

If we multiply (3.22) and (3.23) by $\sigma$ and $z^l$, respectively, and subtract the resulting equations, then after simple calculations we obtain (cf. (3.20))

$$(z^n - \sigma^n) H_n(z) = -\sigma^{n+1} \sum_{k=0}^{n+m} a_{l(n+m+1)+k} z^k$$

$$+ \sigma^{n+1} \sum_{k=l}^{n+m+l+1} a_{l(n+m+1)+k} z^k + G_{n+1}(\rho_1, \sigma, z), \quad (3.24)$$

where $G_n(\rho_1, \sigma, z)$ is given by (3.16). From (3.24), we easily derive (3.21). ∎

## 4. PROOF OF THEOREM 2.1

Since $\sigma > \rho$, $\Delta_{l,n,m}^\sigma(z, f)$ can be estimated from (3.15) to yield

$$|\Delta_{l,n,m}^\sigma(z, f)| \leqslant \frac{C\sigma^n \rho_1^{-ln}}{|z^n - \sigma^n|} \sum_{k=0}^{n+m} \left(\frac{|z|}{\rho_1}\right)^k + \frac{G_n(\rho_1, \sigma, |z|)}{|z^n - \sigma^n|},$$

where $C$ is a positive constant independent of $n$. If we let $|z| = R$, then using (3.16), we obtain

$$|\Delta_{l,n,m}^\sigma(z, f)| \leqslant \frac{C\sigma^n \rho_1^{-ln}}{|R^n - \sigma^n|} \sum_{k=0}^{n+m} \left(\frac{R}{\rho_1}\right)^k$$

$$+ \begin{cases} O\left(\dfrac{\sigma^n}{|R^n - \sigma^n|} \left(\dfrac{R}{\rho_1^{l+2}}\right)^n\right), & \text{if } R \geqslant \rho, \\[3mm] O\left(\dfrac{\sigma^n}{|R^n - \sigma^n|} \left(\dfrac{1}{\rho_1}\right)^{(l+1)n}\right), & \text{if } 0 < R < \rho. \end{cases}$$

A straightforward analysis now gives us

$$\max_{|z| = R} |\Delta_{l,n,m}^\sigma(z, f)| \leqslant C \begin{cases} \sigma^n \rho_1^{-(l+1)n}, & \text{if } R > \sigma, \\[2mm] R^n \rho_1^{-(l+1)n}, & \text{if } \rho \leqslant R < \sigma, \\[2mm] \rho_1^{-ln}, & \text{if } 0 < R < \rho. \end{cases}$$

Since $\varepsilon > 0$ is arbitrary in $\rho_1 = \rho - \varepsilon$, we obtain (cf. (2.1))

$$F_l(R, \sigma) \leqslant \begin{cases} \sigma \rho^{-(l+1)}, & R > \sigma, \\[2mm] R \rho^{-(l+1)}, & \rho \leqslant R < \sigma, \\[2mm] \rho^{-l}, & 0 < R < \rho, \end{cases}$$

that is,

$$F_l(R, \sigma) \leqslant K_l(R, \sigma, \rho), \qquad R > 0, R \neq \sigma. \quad (4.1)$$

In order to prove the reverse inequality, we shall consider two cases (1) $R \geqslant \rho$ and (2) $0 < R < \rho$. Let $\varepsilon > 0$ be so small that

$$\rho^{l+1} < (\rho - \varepsilon)^{l+2} =: \rho_1^{l+2}. \tag{4.2}$$

*Case 1* $(R \geqslant \rho)$. Given integers $l \geqslant 1$ and $m \geqslant -1$, we set for any integer $q$

$$n = \left[\frac{q}{l+1}\right] - m.$$

Then $q$ can be expressed as $q = l(n+m+1) + k_1$, $n+m-l \leqslant k_1 \leqslant n+m$. This shows that $a_q \neq 0$ for some $k_1$ and for infinitely many $n$. If we divide both sides of (3.15) by $z^{k_1+1}$ and then integrate over $|z| = R$, we see on using (3.16) and Cauchy's theorem, that

$$\frac{1}{2\pi i} \int_{|z|=R} \frac{(z^n - \sigma^n) \Delta^\sigma_{l,n,m}(z, f)}{z^{k_1+1}} \, dz$$

$$= a_{l(n+m+1)-n+k_1} - \sigma^n a_{l(n+m+1)+k_1} + O\left(\frac{\sigma^n}{R^{k_1}} \left(\frac{R}{\rho_1^{l+2}}\right)^n\right). \tag{4.3}$$

Since $|z^n - \sigma^n| \leqslant R^n + \sigma^n$, for $|z| = R$, it follows from Definition (2.1) that

$$\left| \int_{|z|=R} \frac{(z^n - \sigma^n) \Delta^\sigma_{l,n,m}(z, f)}{z^{k_1+1}(R^n + \sigma^n)} \, dz \right|$$

$$\leqslant \frac{1}{R^{k_1+1}} \int_{|z|=R} \left\{ \max_{|z|=R} |\Delta^\sigma_{l,n,m}(z, f)| \right\} |dz|$$

$$\leqslant c R^{-k_1} (F_l(R, \sigma) + \varepsilon)^n.$$

Notice that $k_1/n \to 1$ and thus $q/n \to l+1$, as $n \to \infty$. On dividing both sides of (4.3) by $(R^n + \sigma^n)$ and taking the $n$th roots, we get

$$\frac{1}{R}(F_l(R, \sigma) + \varepsilon) \geqslant \varlimsup_{n \to \infty} \left| \frac{a_{q-n} - \sigma^n a_q}{R^n + \sigma^n} + O\left\{ \frac{\sigma^n}{(R^n + \sigma^n) R^{k_1}} \left(\frac{R}{\rho_1^{l+2}}\right)^n \right\} \right|^{1/n}$$

$$= \varlimsup_{n \to \infty} \left| \frac{\sigma^n}{R^n + \sigma^n} a_q \right|^{1/n}.$$

Since

$$\varlimsup_{n \to \infty} \left\{ \frac{\sigma^n}{R^n + \sigma^n} \right\}^{1/n} = \begin{cases} \dfrac{\sigma}{R}, & \text{if } R > \sigma, \\[2mm] 1, & \text{if } R < \sigma, \end{cases}$$

and $\rho_1^{-(l+2)} < \rho^{-(l+1)} = \overline{\lim}_{n\to\infty} |a_q|^{1/n}$, we deduce that

$$\frac{1}{R}(F_l(R,\sigma)+\varepsilon) \geqslant \begin{cases} \dfrac{\sigma}{R}\rho^{-(l+1)}, & \text{if } R > \sigma, \\[2mm] \rho^{-(l+1)}, & \text{if } R < \sigma. \end{cases}$$

In other words,

$$F_l(R,\sigma) \geqslant \rho^{-l} \min\left\{\frac{\sigma}{\rho},\frac{R}{\rho}\right\} = K_l(R,\sigma,\rho), \qquad R \geqslant \rho. \tag{4.4}$$

*Case* 2 $(0 < R < \rho)$. When $0 < R < \rho$, we set for any integer $q$ $n = [q/l] - m - 1$, i.e., $q = l(n+m+1) + k_2$, $0 \leqslant k_2 \leqslant l-1$. Here we observe that $k_2/n \to 0$ and $q/n \to l$, as $n \to \infty$. On following the method used above after (4.3) and taking into account the estimate of $G_n(\rho,\sigma,z)$ when $|z| = R < \rho$, we conclude that

$$F_l(R,\sigma)+\varepsilon \geqslant \overline{\lim_{n\to\infty}} \left|\frac{\sigma^n}{R^n+\sigma^n}a_q\right|^{1/n} = \rho^{-l}.$$

Thus we have

$$F_l(R,\sigma) \geqslant K_l(R,\sigma,\rho). \tag{4.5}$$

From (4.1), (4.4), and (4.5) we obtain the relation (2.3).  ∎

## 5. Proof of Theorem 2.2

First we remark that if for some $z_0$ $\overline{\lim}_{n\to\infty} |\Delta_{l,n,m}^{\sigma}(z_0,f)|^{1/n} < K_l(|z_0|,\sigma,\rho)$ then it follows from (3.20) that

$$\overline{\lim_{n\to\infty}} |H_n(z_0)|^{1/n} < K_l(|z_0|,\sigma,\rho). \tag{5.1}$$

We shall show that there cannot be more than $l$ (or $l-1$) points in $Y_1$ (or $Y_2$) for which (2.5) fails. The proof follows the line of proof of V. Totik [3] (cf. Saff and Varga [4]) and is by contradiction.

*Case* 1 $(|z| > \rho, |z| \neq \sigma)$. Using (3.16), we can rewrite (3.21) as

$$(z^n - \sigma^n)H_n(z) = \sigma^{n+1} \sum_{k=0}^{l} a_{(l+1)(n+m+1)+k} z^{k+n+m+1} + O((\sigma\rho_1^{-l})^n)$$

$$+ O((\sigma|z|\rho_1^{-(l+2)})^n).$$

Next choose $\varepsilon > 0$ satisfying (4.2). Then there exists $\eta > 0$ such that

$$
H_n(z) = \frac{\sigma^{n+1}}{z^n - \sigma^n} \sum_{k=0}^{l} a_{(l+1)(n+m+1)+k} z^{k+n+m+1}
$$
$$
+ \begin{cases} O((\sigma \rho^{-(l+1)} - \eta)^n), & \text{if } |z| < \sigma, \\ O((|z| \rho^{-(l+1)} - \eta)^n), & \text{if } \rho < |z| < \sigma. \end{cases} \tag{5.2}
$$

Assume that

$$
\overline{\lim_{n \to \infty}} |\Delta_{l,n,m}^{\sigma}(z_j, f)|^{1/n} < K_l(|z_j|, \sigma, \rho), \qquad 1 \leqslant j \leqslant l+1,
$$

where $\{z_j\}_{j=1}^{l+1}$ are $l+1$ distinct points in the region $Y_1$.

Without any loss of generality, we may assume that $|z_j| > \sigma$ for $j = 1, ..., \lambda$ and $|z_j| < \sigma$ for $j = \lambda + 1, ..., l+1$. Then, from (5.2), we have

$$
\overline{\lim_{n \to \infty}} |H_n(z_j)|^{1/n} < \begin{cases} \sigma \rho^{-(l+1)}, & \text{if } 1 \leqslant j \leqslant \lambda, \\ |z_j| \rho^{-(l+1)}, & \text{if } \lambda + 1 \leqslant j \leqslant l+1. \end{cases}
$$

This together with (5.2) shows that there are numbers $\eta_1 > 0$ and $C \geqslant 1$ such that for all $n \geqslant 1$

$$
|\beta_{j,n}| < \begin{cases} C(\sigma \rho^{-(l+1)} - \eta_1)^n, & \text{if } 1 \leqslant j \leqslant \lambda, \\ C(|z_j| \rho^{-(l+1)} - \eta_1)^n, & \text{if } \lambda + 1 \leqslant j \leqslant l+1, \end{cases} \tag{5.3}
$$

where

$$
\sum_{k=0}^{l} a_{(l+1)(n+m+1)+k} z_j^k = \frac{z_j^n - \sigma^n}{\sigma^n z_j^{n+m+1}} \beta_{j,n}, \qquad j = 1, 2, ..., l+1. \tag{5.4}
$$

The coefficient matrix in the above system of $(l+1)$ equations is a Vandermondian which is nonsingular since all the $z_j$'s are distinct. Solving the system (5.4) for $a_{\Lambda_n+k}$, where $\Lambda_n := (l+1)(n+m+1)$, we obtain

$$
a_{\Lambda_n+k} = \sum_{j=1}^{l+1} c_{j,k} \frac{z_j^n - \sigma^n}{\sigma^n z_j^{n+m+1}} \beta_{j,n}, \qquad k = 0, 1, ..., l,
$$

where $c_{j,k}$ are constants independent of $n$. Thus from (5.3) we have

$$
\overline{\lim_{n \to \infty}} |a_{\Lambda_n+k}|^{1/[\Lambda_n+k]} \leqslant \max(\xi_1, \xi_2), \qquad 0 \leqslant k \leqslant l,
$$

where

$$
\xi_1 = \overline{\lim_{n \to \infty}} \left\{ \max_{1 \leqslant j \leqslant \lambda} \left| \frac{z_j^n - \sigma^n}{\sigma^n z_j^n} (\sigma \rho^{-(l+1)} - \eta_1)^n \right| \right\}^{1/[\Lambda_n+k]} < 1/\rho
$$

$$
\xi_2 = \overline{\lim_{n \to \infty}} \left\{ \max_{\lambda+1 \leqslant j \leqslant l+1} \left| \frac{z_j^n - \sigma^n}{\sigma^n z_j^n} (|z_j| \rho^{-(l+1)} - \eta_1) \right| \right\}^{1/[\Lambda_n+k]} < 1/\rho,
$$

that is,

$$\varlimsup_{n \to \infty} |a_{(l+1)(n+m+1)+k}|^{1/[(l+1)(n+m+1)+k]} < 1/\rho$$

independently of $0 \leqslant k \leqslant l$. This contradicts the fact that $\varlimsup_{n \to \infty} |a_n|^{1/n} = \rho^{-1}$. Hence the relation (2.6) holds in the region $Y_1$ for all but at most $l$ points.

*Case 2* $(|z| < \rho)$. On using (3.16), we can rewrite (3.21) for $|z| < \rho$ as follows:

$$(z^n - \sigma^n) H_n(z) = -\sigma^{n+1} \sum_{k=0}^{l-1} a_{l(n+m+1)+k} z^k + O(\sigma \rho_1^{-(l+1)})^n)$$
$$+ O((|z| \rho_1^{-(l+1)})^n).$$

Choosing $\rho_1 < \rho$ so close to $\rho$ that $\rho_1^{l+1} > \rho^l$, we get

$$H_n(z) = O((\rho^{-l} - \eta)^n) - \frac{\sigma^{n+1}}{z^n - \sigma^n} \sum_{k=0}^{l-1} a_{l(n+m+1)+k} z^k,$$

where $\eta$ is a sufficiently small positive number. If we assume that

$$\varlimsup_{n \to \infty} |\Delta_{l,n,m}^{\sigma}(z_j, f)|^{1/n} < \rho^{-l}, \qquad 1 \leqslant j \leqslant l,$$

where $\{z_j\}_1^l$ are $l$ distinct points in the region $Y_2$, we arrive at a contradiction on following the procedure used for Case 1.

This completes the proof of Theorem 2.2. ∎

## 6. Proof of Theorem 2.3

(i) We shall first prove the theorem when the set $\{z_j\}$ lies in $Y_1$. Let $z_1, ..., z_l$ be $l$ distinct points with $\sigma \neq |z_j| > \rho$, $1 \leqslant j \leqslant l$. Then the system of $l + 1$ equations

$$\sum_{k=0}^{l} \mu_k z_j^k = 0, \qquad 1 \leqslant j \leqslant l, \tag{6.1}$$

with $\mu_l = 1$ has a unique solution $\mu_0, ..., \mu_{l-1}$. Set

$$f_1(z) := \left( \sum_{k=0}^{l} \mu_k z^k \right) \Big/ \left( 1 - \left( \frac{z}{\rho} \right)^{l+1} \right). \tag{6.2}$$

Then $f_1$ is a rational function with $l+1$ poles on $|z| = \rho$ which implies that $f_1 \in A_\rho$. Notice that

$$
\left(1 - \left(\frac{z}{\rho}\right)^{l+1}\right)^{-1} = \sum_{q=0}^{\infty} \left(\frac{z}{\rho}\right)^{(l+1)q}. \tag{6.3}
$$

If we write $f_1(z) := \sum_{j=0}^{\infty} a_j z^j$, then from (6.2) and (6.3), we find that

$$
a_{(l+1)q+k} = \mu_k \rho^{-(l+1)q}, \qquad 0 \leqslant k \leqslant l,\, q = 0, 1, 2, \dots.
$$

Thus, from (6.1), we have

$$
\sum_{k=0}^{l} a_{(l+1)q+k} z_j^k = \sum_{k=0}^{l} \mu_k \rho^{-(l+1)q} z_j^k = 0, \qquad 1 \leqslant j \leqslant l,\, q = 0, 1, 2, \dots. \tag{6.4}
$$

For any integer $n$, we can determine integers $r$ and $s$ so that $ln + s = (l+1)(r - m - 1)$, where $0 \leqslant s \leqslant l$. More precisely, $r = -[-ln/(l+1)] + m + 1$ (or $r = [ln/(l+1)] + m + 2$, unless $s \equiv 0 \pmod{l+1}$), where $[x]$ denotes the integral part of $x$. That is, $l(n + m + 1) + s + m + 1 = (l+1)r$, $0 \leqslant s \leqslant l$. Consider the decomposition

$$
\sum_{k=0}^{n+m} a_{l(n+m+1)+k} z_j^k = \sum_{k=0}^{s+m} a_{l(n+m+1)+k} z_j^k
$$
$$
+ \sum_{q=r}^{n+m} z_j^{(l+1)q - l(n+m+1)} \sum_{k=0}^{l} a_{(l+1)q+k} z_j^k,
$$

in which the second term on the right side vanishes by (6.3). Therefore,

$$
\sum_{k=0}^{n+m} a_{l(n+m+1)+k} z_j^k = \sum_{k=0}^{s+m} a_{l(n+m+1)+k} z_j^k = O(\rho^{-ln}), \qquad 1 \leqslant j \leqslant l. \tag{6.5}
$$

Since $\sigma \geqslant \rho^2$, from (3.22) and (6.5) we have

$$
(z_j^n - \sigma^n)\, \Delta_{l,n,m}^{\sigma}(z_j, f_1) = O(\sigma^n \rho^{-ln}) + O((\sigma |z_j| \rho_1^{-(l+2)})^n),
$$

where $\rho_1 := \rho - \varepsilon$ satisfies (4.2), i.e.,

$$
\Delta_{l,n,m}^{\sigma}(z_j, f_1) = O\left(\frac{\sigma^n \rho^{-ln}}{|z_j^n - \sigma^n|} + \frac{(\sigma |z_j| \rho_1^{-(l+2)})^n}{|z_j^n - \sigma^n|}\right). \tag{6.6}
$$

Notice that $\rho^{-l} = \rho \rho^{-(l+1)} < |z_j| \rho^{-(l+1)}$, and $\rho_1^{-(l+2)} < \rho^{-(l+1)}$ by (4.2). With this observation, we obtain from (6.6)

$$
\Delta_{l,n,m}^{\sigma}(z_j, f_1) = O\left(\frac{\sigma^n}{|\sigma^n - z_j^n|} \cdot \frac{\delta^n |z_j|^n}{\rho^{(l+1)n}}\right),
$$

where $\delta = \max\{\rho/|z_j|, \ \rho^{l+1}/\rho_1^{l+2}\} < 1$. Hence, for every $j = 1, 2, ..., l$, we get

$$\varlimsup_{n \to \infty} |\Delta_{l,n,m}^{\sigma}(z_j, f_1)|^{1/n} < \begin{cases} \dfrac{\sigma}{\rho^{l+1}}, & \text{if} \ \ |z_j| > \sigma, \\[3mm] \dfrac{|z_j|}{\rho^{l+1}}, & \text{if} \ \ \rho < |z_j| < \sigma. \end{cases}$$

This proves Theorem 2.3 for the region $Y_1$.

(ii) If $z_1, ..., z_{l-1}$ are $l-1$ distinct points with $z_{l-1} \neq 0$ in the region $Y_2$, we solve the system of $l$ equations

$$\sum_{k=0}^{l-1} \mu_k z_j^k = 0, \qquad 1 \leqslant j \leqslant l-1,$$

with $\mu_{l-1} = 1$. Set $f_2(z) := (\sum_{k=0}^{l-1} \mu_k z^k)/(1 - (z/\rho)^l)$. Then, on repeating the above argument with suitable changes, we conclude that

$$\varlimsup_{n \to \infty} |\Delta_{l,n,m}^{\sigma}(z_j, f_2)|^{1/n} < \rho^{-l}, \qquad 1 \leqslant j \leqslant l-1.$$

This completes the proof. ∎

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